

HODGE CYCLES ON SOME MODULI SPACES

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The primary goal of this paper is to verify the Hodge and generalized Hodge conjectures for certain moduli spaces of sheaves over curves and surfaces. When X is a smooth projective curve, let $U_X(n, d)$ be the moduli space of stable vector bundles of rank n and degree d . If n and d are coprime, this space is known to be smooth and projective. del Baño has shown that the Hodge conjecture holds for $U_X(n, d)$ if it holds for all powers of the Jacobian $J(X)$. We reprove this along with an analogous statement for the generalized Hodge conjecture. From this, it is easy to deduce a refinement of result of Biswas and Narasimhan [BN] that the generalized Hodge conjecture is valid for $U_X(n, d)$ when X is very general in moduli. We have some extensions of these results when X is a smooth projective surface. In particular, we show that the Hodge conjecture holds for the moduli space of semistable torsion free sheaves over an abelian or Kummer surface. For arbitrary surfaces, these spaces are difficult to analyze, so our attention is devoted to the rank one case, i.e. the Hilbert scheme of points. We show that the Hodge (respectively generalized Hodge) conjecture holds for the Hilbert scheme of points on X if it holds for all powers of X . For the Hodge conjecture, this last result can be deduced from some work of de Cataldo and Milgiorini [CM1, CM2].

The key reductions in the proofs of these theorems are based on some simple criteria established in the first section. Here we show that the Hodge conjecture holds for a smooth projective variety if can be dominated by, stratified by, or fibered over (with suitable fibers) varieties where the conjecture holds. The second case forces us to deal with the Hodge conjecture for quasiprojective varieties; we use Jannsen's formulation of it. The second section contains a few applications of these ideas apart from the main theorems. For example that the Hodge conjecture holds for a smooth projective variety with \mathbb{C}^* -action if it holds for the fixed point locus. The third section collects some results pertaining to the Hodge conjecture for powers of varieties. These along with the previous lemmas yield the main theorems. In the final section, we consider some arithmetic analogues of these results. We give criteria for the validity of Tate's conjecture for some of the spaces considered above. Also we show, in accordance with a conjecture (or "espoir") of Deligne, that Hodge cycles are absolute on the moduli space of vector bundles over any curve, or on the Hilbert scheme of points over any surface of Kodaira dimension zero.

Except for the last section, all varieties will be defined over \mathbb{C} , and (co)homology groups will be with respect to the usual topology with rational coefficients. Following Kollár, we will refer to a point of the complement of a countable union of proper analytic subvarieties of an analytic variety as a *very general* point. If a variety has an obvious moduli space, we say that the variety is very general if the corresponding point moduli is very general.

Author partially supported by the NSF.

I would like to thank M. de Cataldo, E. Markman, K. Murty, M. Nori, P. Sastry, and Y. Zarhin for providing useful feedback and references. Parts of this paper were written during visits to the University of Chicago in the fall of 2000 and MSRI in the April 2002. My thanks to these institutions for their hospitality.

1. BASIC TOOLS

We first recall some basic notions from Hodge theory. The primary reference for mixed Hodge structures is [D1]. We will refer to a pure rational Hodge structure simply as a Hodge structure. Given a smooth projective variety X and codimension p subvariety $Z \subset X$, it will be convenient to view the fundamental class $[Z]$ as an element of the weight 0 Hodge structure $H^{2p}(X, \mathbb{Q})(p)$. The (p) indicates the Tate twist of the Hodge structure [D1, 2.1.13]; this amounts to shifting the Hodge bigrading by $(-p, -p)$ and modifying the lattice by a factor of $(2\pi i)^p$. Let $H_{alg}^{2p}(X)$ denote the \mathbb{Q} -span of these classes. For any weight 0 Hodge structure H , let H_{hodge} denote the intersection of $H^{0,0}$ with the rational lattice. We will write $H_{hodge}^{2p}(X)$ for $[H^{2p}(X)]_{hodge}$. We always have an inclusion $H_{alg}^*(X) \subset H_{hodge}^*(X)$; the Hodge conjecture asserts the converse.

Next, we recall the generalized Hodge conjecture. This requires some additional notation. The level of a Hodge structure $H = \oplus H^{p,q}$ is the maximum of $\{|p - q| \mid H^{p,q} \neq 0\}$. Given a complex subspace $W \subseteq V$ of a rational Hodge structure, let W_h be largest sub Hodge structure of V contained in W . Note that any sub Hodge structure, such as W_h , is determined by the underlying rational subspace, and we will usually identify it with the subspace. Given a Hodge structure V , we get a filtration $\mathcal{F}^p V = (F^p V)_h$ by sub Hodge structures that we will call the level filtration. If V has weight m , $\mathcal{F}^p V$ is precisely the largest sub Hodge structure of level at most $|m - 2p|$.

Given a smooth projective variety X , the coniveau, or arithmetic filtration, is given by

$$\begin{aligned} N^p H^i(X, \mathbb{Q}) &= \sum_{\text{codim } Y \geq p} \ker[H^i(X, \mathbb{Q}) \rightarrow H^i(X - Y, \mathbb{Q})] \\ &= \sum_{\text{codim } Y = q \geq p} \text{im}[H^{i-2q}(\tilde{Y}, \mathbb{Q})(-q) \rightarrow H^i(X, \mathbb{Q})] \end{aligned}$$

where Y ranges over closed subvarieties; in the second expression $\tilde{Y} \rightarrow Y$ are chosen desingularizations and the maps on cohomology are the Gysin maps. Since the level of $H^{i-2q}(\tilde{Y}, \mathbb{Q})(-q)$ is bounded by $i - 2p$, we have an inclusion

$$N^p H^i(X, \mathbb{Q}) \subseteq \mathcal{F}^p H^i(X, \mathbb{Q})$$

Grothendieck's amended version of the generalized Hodge conjecture $GHC(H^i(X), p)$ asserts that equality holds [Gr2]. We will say that the generalized Hodge conjecture holds for X if $GHC(H^i(X), p)$ is true for all i and p . The space $N^p H^{2p}(X)$ is just the subspace generated algebraic cycles of codimension p , while $\mathcal{F}^p H^{2p}(X) = F^p H^{2p}(X)$ is the space of Hodge cycles i.e. rational (p, p) classes. Hence, $GHC(H^{2p}(X), p)$ is the usual Hodge conjecture.

It will be convenient to define

$$N^p(H^i(X)(c)) = N^{p+c} H^i(X).$$

The notation is chosen so that the inclusion $N^p \subseteq \mathcal{F}^p$ persists after Tate twisting.

We turn now to the functoriality properties of the coniveau filtration.

Proposition 1.1. *The filtrations N^\bullet and \mathcal{F}^\bullet are preserved by*

1. *pushforwards: if $f : X \rightarrow Y$ is a map of smooth projective varieties of dimensions n and m respectively, then*

$$f_*(N^p H^i(X)) \subseteq N^p(H^{i+2(m-n)}(Y)(m-n))$$

2. *pullbacks: if f is as above, then*

$$f^*(N^p H^i(Y)) \subseteq N^p H^i(X),$$

and

3. *products:*

$$N^p(H^i(X)) \otimes N^q(H^j(Y)) \subseteq N^{p+q} H^{i+j}(X \times Y)$$

Proof. An element $t \in N^p H^i(X)$ lies in the image of a map $k_* H^{i-2q}(T)(-q)$ where $k : T \rightarrow X$ is a morphism from a smooth projective variety of dimension $n - q \leq n - p$. Therefore

$$f_*(t) \in (f \circ k)_* H^{i-2q}(T)(m - n - q) \subseteq N^p(H^{i+2(m-n)}(Y)(m - n))$$

This proves the first part.

For the third statement. Let $T \rightarrow X$ and $S \rightarrow Y$ be morphisms from smooth projective varieties such that $\dim T \leq \dim X - p$ and $\dim S \leq \dim Y - q$. Then $\dim T \times S \leq \dim X \times Y - p - q$. It follows that the image of $T \times S$ lies in $N^{p+q}(H^{i+j}(X \times Y))$ as expected.

We now turn to the proof of the second part which is the most involved. To avoid excessive notation, we will suppress Tate twists. To begin with, let us assume that f is surjective. Suppose that the $S \subset Y$ is an irreducible codimension $q \geq p$ subvariety. The preimage $f^{-1}S$ will have codimension less than or equal to q . By taking general hyperplane sections, we can find a cycle $Z \subseteq f^{-1}S$ of codimension exactly q surjecting onto S . By stratification theory [GM, pp. 33-43], we can find a proper Zariski closed set $Z'' \subset Z$ containing the union of singular loci $Z_{\text{sing}} \cup f^{-1}S_{\text{sing}}$, such that the map $f : X - Z'' \rightarrow Y - f(Z'')$ is locally trivial along tubular neighborhoods of $Z' = Z - Z''$ and $S' = S - f(Z'')$. To make the last condition precise, consider the diagram

$$\begin{array}{ccccc}
 N_{Z'} & \xrightarrow{\quad} & X' & & \\
 \downarrow f & \searrow \pi & \nearrow k & & \downarrow f \\
 & & Z' & & \\
 \downarrow f & \swarrow \pi & \downarrow i & & \downarrow f \\
 N_{S'} & \xrightarrow{\quad} & Y' & & \\
 & \searrow \pi & \nearrow i & & \\
 & & S' & &
 \end{array}$$

where $X' = X - Z''$, $Y' = Y - f(Z'')$, and $N_{Z'}$, $N_{S'}$ denotes appropriately chosen tubular neighbourhoods of Z' and S' respectively. The above condition is that $N_{Z'} \rightarrow f^* N_{S'}$ is a locally trivial map of locally trivial fiber bundles (for the classical topology) over Z' . Fiberwise, we have an open immersion of $2q$ real dimensional oriented manifolds, and this induces an isomorphism of compactly supported $2q$

dimensional cohomologies. Thus the Thom class $\tau_{Z'}$ of $N_{Z'}$, which can be viewed element of relative cohomology $H_{Z'}^{2q}(N_{Z'}) = H^{2q}(N_{Z'}, N_{Z'} - Z')$, coincides with the pullback of the Thom class τ'_S on $f^*N_{S'}$. The Gysin map $H^*(Z') \rightarrow H^{*+2q}(X')$ is given by $\alpha \mapsto \pi^*\alpha \cup \tau_{Z'}$ extended by 0 to X' . A similar description holds for (S', Y') . It follows that we have a commutative diagram

$$\begin{array}{ccc} H^*(S') & \longrightarrow & H^{*+2q}(Y') \\ \downarrow & & \downarrow \\ H^*(Z') & \longrightarrow & H^{*+2q}(X') \end{array}$$

Therefore, if

$$\alpha \in \ker[H^*(Y) \rightarrow H^*(Y - S) = H^*(Y' - S')],$$

then its image in $H^*(X)$ maps to

$$\ker[H^*(X) \rightarrow H^*(X - Z) = H^*(X' - Z')].$$

This implies that f^* preserves N^p . The general case is carried out in the same way after replacing S by $S \cap f(X)$. □

A correspondence is an algebraic cycle on $Y \times Z$. Suppose that T is a pure codimension c correspondence which defines an element $[T] \in H^{2c}(Y \times Z)(c)$. This induces a morphism $T_* : H^*(Y) \rightarrow H^{*+2(c-d)}(Z)(c-d)$ given by $\alpha \mapsto p_{Z*}(p_Y^*(\alpha) \cup [T])$, where $d = \dim Y$.

Corollary 1.2. *The action of a correspondence preserves the above filtrations.*

Lemma 1.3. *The operation $V \mapsto \mathcal{F}^p V$ induces an exact functor from the category of polarizable Hodge structures of a fixed weight to itself*

Proof. Let the weight be m . First note that \mathcal{F}^p is a functor: given a morphism $f : V \rightarrow W$, $f(\mathcal{F}^p V)$ is a sub Hodge structure of W lying in $F^p W$, since its level is bounded by $|m - 2p|$. Thus $f(\mathcal{F}^p V)$ lies in $\mathcal{F}^p W$.

Suppose that

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

is an exact sequence of polarizable Hodge structures. By [D1, 2.3.5],

$$0 \rightarrow F^p U \rightarrow F^p V \rightarrow F^p W \rightarrow 0$$

is exact. Certainly, this yields a complex

$$\mathcal{F}^p U \rightarrow \mathcal{F}^p V \rightarrow \mathcal{F}^p W$$

which will be shown to be a short exact sequence. Injectivity of the first map above is clear. The kernel of $\mathcal{F}^p V \rightarrow \mathcal{F}^p W$ is a Hodge structure lying in $F^p U$. Thus it must coincide with $\mathcal{F}^p U$. It remains to check surjectivity of the map $\mathcal{F}^p V \rightarrow \mathcal{F}^p W$. The category of polarizable Hodge structures is semisimple [D1, 4.2.3]. Therefore, there is a splitting $s : W \rightarrow V$ for the morphism $V \rightarrow W$. From the first paragraph we get $s(\mathcal{F}^p W) \subseteq \mathcal{F}^p(V)$. This finishes the proof. □

Putting these results together leads to one of the main tools of this paper.

Corollary 1.4. *Let X and Y be smooth projective varieties and suppose that $f : H^i(X) \rightarrow H^j(Y)((j-i)/2)$ is a surjective morphism of Hodge structures induced by a correspondence (note that $(j-i)/2$ would be an integer). Then $GHC(H^i(X), p)$ implies $GHC(H^j(Y), p + (j-i)/2)$.*

Proof. Let $e = (j-i)/2$, and assume $GHC(H^i(X), p)$. The lemma implies that any element $\alpha \in \mathcal{F}^{p+e}H^j(Y)(e)$ can be lifted to a class $\beta \in \mathcal{F}^pH^i(X) = N^pH^i(X)$. Therefore α lies in N^{p+e} . \square

Next we want to consider the problem of checking the Hodge conjecture for a stratified variety. We will reduce it to the conjecture for strata. This forces us to deal with Hodge cycles on arbitrary quasiprojective varieties. We recall the formulation of these notions due to Jannsen [J]. The appropriate setting for this is homology. For our purposes the Borel-Moore homology $H_i(U)$ of complex algebraic variety U can be taken to be the dual of the compactly supported cohomology $H_c^i(U)$. $H_i(U)$ carries a mixed Hodge structure dual to the one on $H_c^i(U)$ constructed by Deligne [D1]. From this it follows easily that the weights of $H_i(U)$ are concentrated in the interval $[-i, 0]$. This mixed Hodge structure is polarizable in the sense that the associated graded with respect to the weight filtration is polarizable. We define the space of Hodge cycles in $H_{2i}(U)$ to be

$$\begin{aligned} H_{2i}^{hodge}(U) &= Hom(\mathbb{Q}(0), H_{2i}(U)(-i)) \cong Hom(\mathbb{Q}(i), W_{-2i}H_{2i}(U)) \\ &\cong W_{-2i}H_{2i}(U) \cap F^{-i}H_{2i}(U, \mathbb{C}). \end{aligned}$$

Given a closed irreducible i -dimensional subvariety $V \subset U$, there is a fundamental class $[V] \in H_{2i}^{hodge}(U)$. The space of algebraic cycles $H_{2i}^{alg}(U)$ is the span of these classes. The Hodge conjecture for U in degree $2i$ asserts that $H_{2i}^{alg}(U) = H_{2i}^{hodge}(U)$. This can be extended to the generalized Hodge conjecture. We define the niveau filtration by

$$N_p H_i(X, \mathbb{Q}) = \sum_{\dim Y \leq p} im[H_i(Y, \mathbb{Q}) \rightarrow H_i(X, \mathbb{Q})]$$

This is a filtration by submixed Hodge structures. We will only be concerned with its intersection with the pure Hodge structure $W_{-i}H_i(X, \mathbb{Q})$. Fix a compactification \bar{X} . Then an easy argument involving weights shows that

$$N_p W_{-i}H_i(X, \mathbb{Q}) \cong \sum_{\dim Y \leq p} im[H_i(\tilde{Y}, \mathbb{Q}) \rightarrow W_{-i}H_i(\bar{X}, \mathbb{Q})]$$

as \tilde{Y} varies over desingularizations of closed subvarieties. This together with Poincaré duality:

$$H_i(\tilde{Y}, \mathbb{Q}) \cong H^{2\dim Y - i}(\tilde{Y}, \mathbb{Q})(\dim Y)$$

gives an estimate on the level which shows

$$N_p W_{-i}H_i(X, \mathbb{Q}) \subseteq \mathcal{F}^{-p}W_{-i}H_i(X, \mathbb{Q}) = \mathcal{F}_p W_{-i}H_i(X, \mathbb{Q})$$

(where \mathcal{F}_p is defined by the last equality). Following Jannsen [J] and Lewis [L], we say that $GHC(H_i(X), p)$ holds if equality holds above. Note that if X is smooth and projective of dimension n , then $GHC(H_i(X), p)$ is equivalent to $GHC(H^{2n-i}(X), n-p)$ by Poincaré duality. It turns out that the generalized Hodge conjecture in this setting is no stronger than in the usual formulation.

Proposition 1.5. *If X' is a desingularization of a compactification of X , then $GHC(H_i(X'), p)$ implies $GHC(H_i(X), p)$.*

The proof hinges on natural isomorphism

$$W_{-i}H_i(X, \mathbb{Q}) \cong H_{-i}(X', \mathbb{Q}),$$

see [J, 7.9] and [L, appendix A].

Note that H_* is functorial in two different ways. It is covariant for proper maps, and contravariant for open immersions. These are dual to the proper restriction and extension by zero maps on H_c^* . It follows that the mixed Hodge structure on H_* is compatible with both kinds of maps. It is also easy to see that the niveau filtration is also compatible with these.

Lemma 1.6. *Let $Z \subset X$ be a closed subset of a projective variety X , and let $U = X - Z$. Then $GHC(H_i(U), p)$ and $GHC(H_i(Z), p)$ imply $GHC(H_i(X), p)$.*

Proof. The exact sequence

$$H_i(Z) \rightarrow H_i(X) \rightarrow H_i(U) \rightarrow H_{i-1}(Z) \dots$$

is compatible with mixed Hodge structures because it is dual to the long exact sequence for the cohomology of a pair. Since $H \mapsto W_*H$ is exact on the category of mixed Hodge structures [D1, 2.3.5], we get an exact sequence of pure polarizable Hodge structures

$$W_{-i}H_i(Z) \rightarrow W_{-i}H_i(X) \rightarrow W_{-i}H_i(U).$$

Lemma 1.3 implies that

$$\mathcal{F}_p W_{-i}H_i(Z) \xrightarrow{f} \mathcal{F}_p W_{-i}H_i(X) \xrightarrow{g} \mathcal{F}_p W_{-i}H_i(U)$$

is exact.

Suppose that $\alpha \in \mathcal{F}_p W_{-i}H_i(X)$. The hypothesis implies that $g(\alpha) \in N^p W_{-i}H_i(U)$. Thus $g(\alpha)$ is the image $\beta \in W_{-i}H_i(T)$ under the map on homology induced by a proper map $T \rightarrow U$, where T is a finite union of varieties of dimension $\leq p$. Choose a compactification \bar{T} of T . By blowing up, if necessary, we can assume that there is a map $\bar{T} \rightarrow X$ extending $T \rightarrow U$. As above, we have a surjection

$$W_{-i}H_i(\bar{T}) \rightarrow W_{-i}H_i(T).$$

Let β' be an element of the space on left lifting β . Then β' maps to an element $\alpha' \in N_p W_{-i}H_i(X)$ such that the difference $\alpha - \alpha'$ lies in the kernel of g , and hence is given by $f(\gamma)$ where $\gamma \in \mathcal{F}_p W_{-i}H_i(Z)$. By hypothesis $\mathcal{F}_p W_{-i}H_i(Z) = N_p W_{-i}H_i(Z)$. Therefore $\alpha = \alpha' + \gamma \in N_p W_{-i}H_i(X)$. □

By a stratification of an algebraic variety X , we will mean a finite partition $X = \cup X_i$ into locally closed sets, called strata, such that closure of any stratum is a union of strata.

Corollary 1.7. *Suppose that X is a projective variety. If X has a stratification such that each stratum S satisfies $GHC(H_i(S), p)$, then $GHC(H_i(X), p)$ holds.*

Proof. Since a stratum of minimal dimension is closed, the result follows by induction on the number of strata. □

Lemma 1.8. *Let $X = Y \times F$ such that $H_*(F)$ is spanned by algebraic cycles. Then $GHC(H_{i-2j}(Y), p-j)$ for all j , such that $i - 2j \geq 0$ and $H_{2j}(F, \mathbb{Q}) \neq 0$, implies $GHC(H_i(X), p)$.*

Proof. The Künneth formula implies that

$$H_k(X) = \bigoplus_{i+2j=k} H_i(Y) \otimes H_{2j}(F)$$

since by assumption the odd degree homology of F vanishes. We can choose a basis of $H_{2j}(F)$ consisting of fundamental classes of varieties. These classes are pure of type $(-j, -j)$, so $H_{2j}(F)$ is a sum of $\mathbb{Q}(j)$'s. Since the Künneth decomposition respects mixed Hodge structures, we have

$$\mathcal{F}_p W_{-k} H_k(X) = \bigoplus_{i+2j=k} \mathcal{F}_{p-j} W_{-i} H_i(Y) \otimes H_{2j}(F)$$

By assumption, the right hand sum equals

$$\bigoplus_{i+2j=k} N_{p-j} W_{-i} H_i(Y) \otimes N_j W_{-j} H_{2j}(F) \subseteq N_p W_{-k} H_k(X)$$

□

Lemma 1.9. *Let $f : X \rightarrow Y$ be a morphism of smooth varieties which is a Zariski locally trivial fiber bundle with fiber F . Suppose that F is smooth and that $H_*(F)$ is spanned by algebraic cycles. Then $GHC(H_{i-2j}(Y), p-j)$ for all j , such that $i-2j \geq 0$ and $H_{2j}(F, \mathbb{Q}) \neq 0$, implies $GHC(H_i(X), p)$.*

Proof. The argument is similar to the proof of the previous lemma, however we will work in cohomology. This is justified by the fact that our spaces are smooth. Let β'_l be algebraic cycles in F whose fundamental classes give a basis for $H^*(F)$. Choose a Zariski open set $U \subset Y$ such that $f^{-1}U \rightarrow U$ is a product. Let β_l be the fundamental classes of the closures in X of the pullbacks of the β'_l to U . These classes define a splitting of the restriction $H^*(X) \rightarrow H^*(F)$, so we can apply the Leray-Hirsch theorem [Sp, p. 258] to obtain a decomposition

$$H^k(X) = \bigoplus_{i+2j=k} H^i(Y) \otimes H^{2j}(F).$$

The rest of the proof proceeds exactly as before. □

Corollary 1.10. *Let $f : X \rightarrow Y$ be a morphism of smooth varieties which is a (Zariski) locally trivial fiber bundle with fiber F . Suppose that F is smooth and that $H_*(F)$ is spanned by algebraic cycles. Then the Hodge conjecture holds for X if it holds for Y .*

Remark 1.11. *A variety has a cellular decomposition if it has a stratification such that the strata are affine spaces. If F has a cellular decomposition, then it satisfies the hypothesis of the lemma [F1, 19.1.11]. Flag manifolds and nonsingular projective toric varieties have cellular decompositions.*

We need a variation on corollary 1.4.

Lemma 1.12. *Suppose that $f : X \rightarrow Y$ is a proper morphism of varieties such that $f_* : H_k(X) \rightarrow H_k(Y)$ is surjective. Then $GHC(H_k(X), p)$ implies $GHC(H_k(Y), p)$.*

Proof. Assume $GHC(H_k(X), p)$. As in the proof of lemma 1.6, we get a surjection of pure polarizable Hodge structures $W_{-k} H_k(X) \rightarrow W_{-k} H_k(Y)$. Consequently, lemma 1.3 implies that $N_p W_{-k} H_k(X) = \mathcal{F}_p W_{-k} H_k(X)$ surjects onto $\mathcal{F}_p W_{-k} H_k(Y)$. So $GHC(H_k(Y), p)$ follows. □

We will give a simple criterion for surjectivity of f_* . Recall that a locally compact Hausdorff space X is a rational homology manifold of (real) dimension n if for each $x \in X$, the rational homology groups $H_*(X, X - \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. For such a space the groups $H_n(X, X - \{x\})$ form a local system as x varies. The space is orientable if this local system is constant. Orientable rational homology manifolds satisfy Poincaré duality with rational coefficients. The most interesting examples of orientable rational homology manifolds for us are varieties with at worst finite quotient singularities.

Lemma 1.13. *If $f : X \rightarrow Y$ is a proper surjective map of quasiprojective varieties such that Y is also a rational homology manifold, then $f_* : H_*(X) \rightarrow H_*(Y)$ is surjective.*

Remark 1.14. *The properness is necessary to insure that f_* is defined.*

Proof. Suppose that X is smooth and that $\dim X = \dim Y$. Then Poincaré duality gives a map $f^* : H_*(Y) \rightarrow H_*(X)$ such that $f_* f^* = \deg(f)$, and this implies surjectivity of f_* .

In general, an intersection of a finite number of very ample divisors will produce a subvariety $X' \subseteq X$ such that $\dim X' = \dim X$ and $f|_{X'}$ is surjective. Let $X'' \rightarrow X$ be a desingularization, and let $g : X'' \rightarrow Y$ be the natural map. Then g_* is surjective as above, but it factors through f_* , so we are done. \square

Corollary 1.15. *With the notation of the lemma, if X satisfies the Hodge (respectively generalized Hodge) conjecture then so does Y .*

2. SOME SIMPLE EXAMPLES

Let us work out some examples to illustrate the previous techniques.

Lemma 2.1. *Let X be a smooth projective variety, and let $V \subset X$ be a smooth closed subvariety. If X and V satisfy the Hodge conjecture, then so does the blow up $\pi : \tilde{X} \rightarrow X$ of X along V .*

Proof. Let $U = X - V$ and let $E = \pi^{-1}V$. $E \rightarrow V$ is a locally trivial bundle with projective spaces as fibers. Therefore E satisfies the Hodge conjecture by lemma 1.9. U satisfies the conjecture by proposition 1.5. Therefore, we are done by lemma 1.6. \square

Corollary 2.2. *The Hodge conjecture is birationally invariant in dimensions up to 5. In other words, if X and X' are two birational smooth projective varieties with dimension ≤ 5 , then one of them satisfies the Hodge conjecture if and only if the other does.*

Proof. Suppose that X satisfies the Hodge conjecture. It is well known that the Hodge conjecture holds in dimensions up to 3 (by the Lefschetz (1,1) and hard Lefschetz theorems). Thus the conjecture holds for a single blow up of X , since the center has dimension ≤ 3 . By iteration, the same goes for any $X'' \rightarrow X$ given by a sequence of blow ups. Since any X' birational to X is dominated by such an X'' , the result follows by corollary 1.15. \square

Lemma 2.3 (Conte-Murre [CN]). *A projective uniruled 4-fold satisfies the Hodge conjecture.*

Proof. Suppose X is a projective uniruled 4-fold. Then any desingularization is also uniruled. Therefore we can assume that X is smooth by proposition 1.5. There exists a surjective map $X' \rightarrow X$ where X' is a smooth projective variety birational to the product of \mathbb{P}^1 and a smooth threefold. Therefore X' satisfies the Hodge conjecture by lemma 1.9 and corollary 2.2. Consequently the conjecture holds for X by corollary 1.15. \square

Lemma 2.4. *Suppose X is a smooth projective variety on which an algebraic torus $T = (\mathbb{C}^*)^N$ acts. X satisfies $GHC(H_i(X), p)$ if every component $S \subset X^T$ of the fixed point set satisfies $GHC(H_i(S), p)$. In particular, X satisfies the Hodge (respectively generalized Hodge) conjecture if the components of the fixed point locus X^T do.*

Proof. Let $\mathbb{C}^* \rightarrow T$ be given by $t \mapsto (t^{a_1}, t^{a_2}, \dots)$. By choosing $1 \ll a_1 \ll a_2 \ll \dots$, we can arrange equality of the fixed point sets $X^T = X^{\mathbb{C}^*}$. Thus we can assume that $T = \mathbb{C}^*$. By a theorem of Bialynicki-Birula [BB], X has a stratification such that the strata are affine space bundles over components of the fixed point locus. The result follows from corollary 1.7 and lemma 1.9. \square

Corollary 2.5. *The (generalized) Hodge conjecture holds if $\dim X^T \leq 3$ (≤ 2).*

When the fixed points are isolated, Bialynicki-Birula's decomposition is a cellular decomposition, so all the cohomology is algebraic.

3. HODGE CONJECTURE FOR PRODUCTS

In this section, we give some criteria for the Hodge conjecture to hold for a power or self product of a variety X . The basic tool for this is the Mumford-Tate group of a Hodge structure. The main reference is [DMOS, I, sect. 3, sect. 5], although the summary in [Grd, II] should be sufficient and a bit more accessible. Let us start with an abstract characterization since it explains the significance most clearly. Given a collection of rational Hodge structures H_i , let $\langle H_i \rangle$ be the tensor H_i . Tannakian considerations show that $\langle H, \mathbb{Q}(1) \rangle$ is equivalent to the category of rational representations of a canonically determined algebraic group defined over \mathbb{Q} ; this is the Mumford-Tate group $MT(H)$. By definition, sub-Hodge structures of $H^{\otimes n}$ are the same thing as $MT(H)$ -submodules. There is a homomorphism $\tau : MT(H) \rightarrow MT(0) = \mathbb{G}_m$ induced by the inclusion $\langle \mathbb{Q}(1) \rangle \subset \langle H, \mathbb{Q}(1) \rangle$. The Hodge or special Mumford-Tate group $Hdg(H)$ is the kernel of this map. The key property which makes this notion useful is that the $Hdg(H)$ invariant tensors in $H^{\otimes n}$ are exactly the Hodge cycles in this space ($MT(H)$ will act on the space of Hodge cycles through a power of τ). An alternative characterization is as follows: the real Hodge structure $H \otimes \mathbb{R}$ has an action of \mathbb{C}^* (viewed as a real group). $Hdg(H)$ can be defined as the smallest \mathbb{Q} -algebraic subgroup of $GL(H)$ whose real points contain the image of $U(1) \in GL(H \otimes \mathbb{R})$. When H has a polarization ψ (which is a symmetric or alternating form or according to the weight), then $Hdg(H)$ is a reductive subgroup of $SO(\psi)$ or $Sp(\psi)$.

The best understood (nontrivial) case is that of an abelian variety X . We refer the reader to the survey articles [Grd, Mu2] for further details and references. In this case, let $Hdg(X) = Hdg(H^1(X))$. Given a polarization ψ of X , the Lefschetz group $Lef(X)$, is the centralizer of $End(X) \otimes \mathbb{Q}$ in $Sp(H^1(X), \psi)$. The Lefschetz

group turns out to be independent of the polarization, and it always contains the Hodge group.

Recall by Poincaré reducibility [Mu], any abelian variety is isogenous to a product of simple abelian varieties. Let X be a simple abelian variety, then $E = \text{End}(X) \otimes \mathbb{Q}$ will be a division algebra over its center K which is a number field. Albert [loc. cit., p 201] has classified the possibilities:

- I $E = K$ is a totally real field.
- II K is totally real, and E is a totally indefinite quaternion algebra over K .
- III K is totally real, and E is a totally definite quaternion algebra over K .
- IV K is a CM field.

Theorem 3.1 (Murty, Ribet). *If X is an abelian variety, then $\text{Hdg}(X) = \text{Lef}(X)$ if and only if all Hodge classes on all powers of X are products of divisor classes. In particular, equality of these groups implies that the Hodge conjecture holds for all powers of X .*

Remark 3.2. *There are a number of interesting cases where these conditions are satisfied. $\text{Hdg}(X) = \text{Lef}(X)$ when:*

1. X is a simple abelian variety with $\dim X$ an odd prime (Tankeev, Ribet [R]),
2. X an abelian variety of dimension 2 or 3 (this appears to be part of the standard folklore; a proof can be found in [MZ]).
3. $E = \text{End}(X) \otimes \mathbb{Q}$ is a totally real number field such that $\dim X/[E : \mathbb{Q}]$ is odd [loc. cit.] (it follows that X is simple of type I),
4. E is a CM field such that $\dim X/[E : \mathbb{Q}]$ is prime [loc. cit.] (in particular, X is simple of type IV),
5. X is isogenous to a product of elliptic curves (see Murty [Mu1]),
6. X is the Jacobian of a very general curve (corollary 3.11),
7. X is a very general abelian variety. (This essentially goes back to Mattuck; a proof can be given along the lines of that of corollary 3.11.)
8. The Jacobian of a modular curve $X_1(N)$ which is the compactification of the quotient of the upper half plane by

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0, a \equiv 1 \pmod{N} \right\}$$

(Hazama, Murty)

Corollary 3.3. *The Hodge conjecture holds for all powers of X , provided that it is on the above list.*

Another class of examples where the conclusion of the corollary is known, even though the Lefschetz and Hodge groups may differ, comes from the work of Shioda [Sh].

Theorem 3.4. *If J is the Jacobian of the Fermat curve $x^m + y^m + z^m = 0$ where m is prime or $m \leq 20$, then the Hodge conjecture holds for all powers of J ,*

Proof. This is an immediate consequence of [Sh, 4.4]. □

Hazama [Hz] has established the generalized Hodge conjecture in certain of the above cases [loc. cit., pg 201]. The theorem can be formulated as follows:

Theorem 3.5 (Hazama). *Let X be an abelian variety satisfying $Hdg(X) = Lef(X)$ and such that all simple factors are of types I or II, then the generalized Hodge conjecture holds for X .*

Corollary 3.6. *If X is as above, then the generalized Hodge conjecture holds for all powers of X .*

Proof. $Hdg(X^k) = Hdg(X)$ since $H^1(X)$ and $H^1(X^k) = H^1(X)^k$ generate the same tensor category. Also $Lef(X) = Lef(X^k)$ [Mi, cor. 4.7]. Therefore X^k satisfies the same conditions as the theorem. \square

Corollary 3.7. *If $E = End(X) \otimes \mathbb{Q}$ is a totally real number field such that $\dim X/[E : \mathbb{Q}]$ is odd then the generalized Hodge conjecture holds for all powers X .*

Proof. The conditions imply that X is simple of type I. Also by remark 3.2, $Hdg(X) = Lef(X)$. \square

The corresponding results for curves follows from:

Proposition 3.8. *The Hodge (respectively generalized Hodge) conjecture holds for all powers of a smooth projective curve X if and only if it holds for all powers of its Jacobian $J(X)$.*

Proof. Suppose that the (generalized) Hodge conjecture holds for $J(X)^k$. Choose a base point on X , and let $\alpha : X \rightarrow J(X)$ be the corresponding Abel-Jacobi map. Then $\alpha^* : H^*(J(X)) \rightarrow H^*(X)$ is a surjection. The Künneth formula implies that the induced map $\alpha^k : X^k \rightarrow J(X)^k$ also induces a surjection on cohomology. Therefore the (generalized) Hodge conjecture holds for X^k by corollary 1.4.

Let g be the genus of X . Consider the map $\beta : X^g \rightarrow J(X)$ given by $(x_1, \dots, x_g) \mapsto \alpha(x_1) + \dots + \alpha(x_g)$. This map induces a surjection $\beta_*^k : H^*(X^{gk}, \mathbb{Q}) \rightarrow H^*(J(X)^k, \mathbb{Q})$ by lemma 1.13. Thus the (generalized) Hodge conjecture for X^{gk} implies it for $J(X)^k$. \square

Define $Hdg(X) = Hdg(J(X))$ and $Lef(X) = Lef(J(X))$. Then:

Corollary 3.9. *If $Hdg(X) = Lef(X)$ or if X is Fermat of degree $m \leq 20$ or a prime, then the Hodge conjecture holds for all powers of X .*

The characterization of Mumford-Tate groups [DMOS, p. 43] together with [D2, 7.5] (see also [Sc, 2.2-2.3]) yields:

Lemma 3.10. *Given a polarized integral variation of Hodge structure V over a smooth irreducible complex variety T , there exists a countable union of proper analytic subvarieties $S \subset T$ such that $Hdg(V_t)$ contains a finite index subgroup of the monodromy group*

$$image[\pi_1(S, t) \rightarrow GL(V_t)]$$

for $t \notin S$.

Corollary 3.11. *If X is very general in the moduli space of curves, then the generalized Hodge conjecture holds for all powers of X .*

Proof. Choose $n \geq 3$ and let $M_{g,n}$ be the fine moduli space of smooth projective curves of genus g with level n structure [AO, 13.4]. Let $\pi : \mathcal{X} \rightarrow M_{g,n}$ be the universal curve. Lemma 3.10 applied to $\mathbb{R}^1\pi_*\mathbb{Z}$ shows that there exist a countable union of proper subvarieties $S' \subset M_{g,n}(\mathbb{C})$ such that a finite index subgroup of the monodromy group

$$\Gamma = \text{image}[\pi_1(M_{g,n}, t) \rightarrow GL(H^1(\mathcal{X}_t))]$$

is contained in $Hdg(\mathcal{X}_t)$ for each $t \notin S'$. Let S be the image of S' in $M_g(\mathbb{C})$. By Teichmüller theory, any finite index subgroup of Γ is seen to be Zariski dense in the symplectic group (see [Ha, 12]). Hence the Hodge group contains the symplectic group whenever $t \notin S$. But this forces

$$Hdg(\mathcal{X}_t) = Lef(\mathcal{X}_t) = Sp(H^1(\mathcal{X}_t)).$$

Fix $X = \mathcal{X}_t$, with t as above. We will show that $\text{End}(X) \otimes \mathbb{Q} = \mathbb{Q}$, and this will finish the proof by corollary 3.7. The natural map

$$\text{End}(X) \otimes \mathbb{Q} \rightarrow \text{End}(H^1(X, \mathbb{Q}))$$

is injective, and the image lies in the ring $\text{End}_{MHS}(H^1(X))$ of endomorphisms of the Hodge structure $H^1(X)$. This is contained in the space $Hdg(X)$ -equivariant maps automorphisms of $H^1(X)$. $Hdg(X)$ acts irreducibly, since it is the full symplectic group. Therefore Schur's lemma implies that $\text{End}(X) \otimes \mathbb{Q} = \mathbb{Q}$ as claimed. \square

Given a smooth projective surface X , $H^2(X, \mathbb{Z})$ carries a symmetric bilinear form \langle, \rangle given by cup product. Let $A(X) = H_{alg}^2(X, \mathbb{Q})$ or equivalently the Neron-Severi group tensored with \mathbb{Q} . The rational transcendental lattice $T(X)$ is the orthogonal complement $A(X)^\perp$. The decomposition

$$H^2(X, \mathbb{Q}) = A(X) \oplus T(X)$$

respects Hodge structure.

Lemma 3.12. *$T(X)$ is the smallest rational Hodge substructure containing $H^{20}(X)$.*

Proof. Let V be the smallest Hodge substructure containing $H^{20}(X)$. Then V^\perp is a rational subspace lying in $H^{11}(X)$. Therefore $V^\perp \subseteq A(X)$ by the Lefschetz (1, 1) theorem, and this implies $T \subseteq V$. On the other hand, the Hodge-Riemann bilinear relations imply that $H^{20}(X)$ is orthogonal to $H^{11}(X)$. Therefore $H^{20}(X) \subseteq T(X)$, and this give the opposite inclusion. \square

If $Z \subset Y \times X$ is a codimension 2 correspondence, then the induced morphism $Z_* : H^i(Y, \mathbb{Q}) \rightarrow H^i(X, \mathbb{Q})$ can be restricted to H^{i0} to get a map $H^0(\Omega_Y^i) \rightarrow H^0(\Omega_X^i)$. This can be interpreted directly in terms of differential forms. After, replacing Z by a resolution \tilde{Z} , the map is a composition of the pullback $H^0(\Omega_Y^i) \rightarrow H^0(\Omega_{\tilde{Z}}^i)$ and the trace $H^0(\Omega_{\tilde{Z}}^i) \rightarrow H^0(\Omega_X^i)$. By definition Z_* preserves A , and it preserves T by lemma 3.12.

Proposition 3.13. *Let X and Y be smooth projective surfaces, and let $Z \subset Y \times X$ be a codimension 2 correspondence. The maps*

$$Z_* : H^0(Y, \Omega_Y^i) \rightarrow H^0(X, \Omega_X^i), \quad i = 1, 2$$

are surjective if and only if

$$Z_* : H^1(Y) \rightarrow H^1(X) \text{ and } Z_* : T(Y) \rightarrow T(X)$$

are surjective. Suppose that these conditions hold and that Y^k satisfies the Hodge (respectively generalized Hodge) conjecture for all $k \leq n$, then X^k satisfies the Hodge (respectively generalized Hodge) conjecture for all $k \leq n$.

Before giving the proof, note that the symmetric group S_n acts on X^n , and hence on $H^*(X^n)$ by permutation of factors. This action is compatible with the Künneth decomposition in the following sense: given classes $\alpha_j \in H^*(X)$

$$\sigma(\alpha_1 \times \alpha_2 \times \dots \times \alpha_n) = \alpha_{\sigma(i_1)} \times \alpha_{\sigma(i_2)} \times \dots \times \alpha_{\sigma(i_n)}$$

Proof. The equivalence of the surjectivity statements follows from elementary Hodge theory and lemma 3.12. It is enough to prove the remainder of the proposition for $k = n$. The correspondence $Z^n \subset Y^n \times X^n$ induces morphisms $H^i(Y^n) \rightarrow H^i(X^n)$. This is compatible with the Künneth decompositions

$$\begin{aligned} H^i(Y^n) &= \bigoplus_{j_1, j_2, i_1, \dots, \sigma} \sigma[N(Y)^{\otimes j_1} \otimes T(Y)^{\otimes j_2} \otimes H^{i_1}(Y) \otimes \dots \otimes H^{i_{n-1}}(Y)] \\ H^i(X^n) &= \bigoplus_{j_1, j_2, i_1, \dots, \sigma} \sigma[N(X)^{\otimes j_1} \otimes T(X)^{\otimes j_2} \otimes H^{i_1}(X) \otimes \dots \otimes H^{i_{n-1}}(X)], \end{aligned}$$

where $i_\ell \neq 2$, $2j_1 + 2j_2 + \sum i_\ell = i$ and σ ranges over a set of permutations. By hypothesis, any Hodge cycle in

$$N(X)^{\otimes j_1} \otimes T(X)^{\otimes j_2} \otimes H^{i_1}(X) \otimes \dots$$

can be lifted to a product of an algebraic cycle with a Hodge cycle on $H^{i-2j_1}(Y^{n-j_1}) \dots$

□

Proposition 3.14. *Let X be a smooth projective surface, then:*

- a) *If $f : Y \dashrightarrow X$ is a dominant rational map of smooth projective surfaces, X^k satisfies the Hodge (respectively generalized Hodge) conjecture for all $k \leq n$ if Y^k does for all $k \leq n$. In particular, the condition is birationally invariant.*
- b) *If X is a rational surface, then X^k satisfies the generalized Hodge conjecture for all $k > 0$.*
- c) *If $X \rightarrow C$ is a (possibly nonminimal) ruled surface, then X^k satisfies the Hodge (respectively generalized Hodge) conjecture for all $k \leq n$ if it holds for all C^k in the same range. In particular, the Hodge conjecture holds for all k if $\text{Hdg}(C) = \text{Lef}(C)$ or C is Fermat of degree ≤ 20 or a prime.*
- d) *If X is an abelian surface, then X^k satisfies the Hodge conjecture for all k .*
- e) *If X is a K3 surface such that $(T(X), \langle, \rangle)$ can be embedded isometrically in $H^{\oplus 3}$, where $H = \mathbb{Q}^2$ with the quadratic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then X^k satisfies the Hodge conjecture for all k .*
- f) *If X is an Kummer surface or a K3 surface with Picard number ≥ 19 , then X^k satisfies the Hodge conjecture for all k .*

Proof. a) Let $f : Z \rightarrow Y$ be a sequence of blow ups such that $Y \dashrightarrow X$ extends to a morphism $g : Z \rightarrow X$. f^* induces isomorphisms $H^1(Y) \rightarrow H^1(Z)$ and $T(Y) \cong T(Z)$, and g_* induces surjections $H^1(Z) \rightarrow H^1(X)$ and $T(Z) \rightarrow T(X)$. Therefore a) follows from proposition 3.13.

b) By a), we can assume that $X = \mathbb{P}^2$. The statement is clear since X^n has a cellular decomposition.

c) By a), we can assume that X is minimal. The statement follows from lemma 1.9 and corollary 3.9.

d) X is either simple or isogenous to a product of two elliptic curves. So the result follows from corollary 3.3.

e) By a theorem of Mukai [Mk2, 1.12], there is an abelian surface A and a correspondence on $X \times A$ inducing an isomorphism $T(A) \cong T(X)$. Therefore the result follows from d) and proposition 3.13.

f) Follows from e) since the conditions imply that $T(X)$ embeds into H^3 . \square

Finally, we give some criteria for the Hodge conjecture to hold when the Hodge group is large.

Theorem 3.15. *Suppose that Y is a smooth projective variety such that $d = \dim Y$ is odd and*

1. $H^i(Y) = 0$ when i is odd and different from d
2. $H^i(Y) = H_{\text{alg}}^i(Y)$ when i is even.
3. $\text{Hdg}(H^d(Y))$ coincides with the symplectic group $\text{Sp}(H^d(Y))$ with respect to the cup product.

Then the Hodge conjecture holds for all powers of Y .

A somewhat weaker analogue can be proven for even dimensional varieties. As noted earlier, the symmetric group S_m acts on $H^i(Y^m)$. Call an element $\alpha \in H^i(Y^m)$ antisymmetric if $\sigma(\alpha) = (-1)^\sigma \alpha$, where $(-1)^\sigma$ denotes the parity of σ .

Theorem 3.16. *Suppose that Y is a smooth projective variety such that $d = \dim Y$ is even and*

1. $H^i(Y) = 0$ when i is odd
2. $H^i(Y) = H_{\text{alg}}^i(Y)$ when i is even and different from d .
3. $\text{SO}(T) \subseteq \text{Hdg}(H^d(Y))$ for $A \subseteq H_{\text{alg}}^d(Y)$ (equality is not required) and $T = A^\perp \subset H^d(Y)$.

Then there exists an antisymmetric Hodge class $\delta \in H^{d\tau}(Y^\tau)$, where $\tau = \dim T$, such that any Hodge class on a Y^n is a linear combination $\alpha + \sum \sigma_i(\delta \times \beta_i)$ where α and β_i are algebraic cycles, and $\sigma_i \in S_n$.

The proof of these theorems is an exercise in invariant theory. Let V be a d -dimensional vector space over a field of characteristic 0 with a nondegenerate alternating or symmetric bilinear form ψ . Let $G = \text{Sp}(V, \psi)$ in the first case, and let $G = \text{SO}(V, \psi)$ in the second. The form induces an isomorphism $V \cong V^*$ as G modules. Therefore ψ can be regarded as a tensor in $V \otimes V$ which is invariant under the G . Choose a nonzero element $\delta \in \wedge^d V$ which we identify with an antisymmetric tensor in $V^{\otimes d}$. This is also G -invariant. The symmetric group S_N acts on $V^{\otimes N}$ by permuting factors, and this action commutes with the G action. Tensor products of the previous tensors, and their transforms under the symmetric group, generate all G -invariant tensors. More precisely:

Lemma 3.17 (Weyl). *With the above notation*

1. $V^{\otimes n}$ is spanned by the S_n orbit of $\{\psi^{\otimes n/2}\}$ if $G = \text{Sp}(V, \psi)$, and
2. $V^{\otimes n}$ is spanned by the union of the S_n orbits of $\{\psi^{\otimes n/2}\}$ and $\{\delta \otimes \psi^{\otimes (n-d)/2}\}$ otherwise;

where these sets are taken to be empty unless the exponents are nonnegative integers.

Proof. See [FH, F.13, F.15] \square

Proof of Theorems 3.15 and 3.16. The proofs of both theorems will run mostly in parallel with occasional branching. In the case of theorem 3.16 we can assume that $\dim T > 1$, otherwise the theorem is vacuous. Since T can be identified with $H^d(Y)/A$, it carries a natural Hodge structure. Set $V = H^d(Y)$. If d is odd, let ψ denote the cup product form on V , and let $G = Sp(V, \psi)$. If d is even, let ψ denote the restriction of the cup product form to T , and let $G = SO(T, \psi)$. By hypothesis $Hdg(V)$ contains G . Therefore, the Hodge cycles in $V^{\otimes i}(j)$ are G -invariant. Since T is irreducible and nontrivial, this proves theorem 3.16 for the first power Y^1 .

We will show that the class ψ is represented by an algebraic cycle on $Y \times Y$. In this paragraph we treat the case where d is odd. By Künneth's formula we have

$$H^{2d}(Y \times Y) = [V \otimes V] \oplus W$$

where

$$W = \bigoplus_{i \neq d} [H^i(Y) \otimes H^{2d-i}(Y)].$$

Then by the first two assumptions of theorem 3.15, $W \subset H_{alg}^{2d}(Y \times Y)$. In particular, the sum Δ' of the Künneth components of the diagonal Δ in W is algebraic. By lemma 3.17, $[V \otimes V]^G$ is one dimensional. Therefore it must be spanned by $\Delta - \Delta'$. In particular, the form ψ in $V \otimes V$ is algebraic.

Now suppose d is even and $\dim T \geq 2$. We decompose $V \otimes V$ further as $[T \otimes T] \oplus V_1 \oplus V_2$ where $V_1 = [T \otimes A] \oplus [A \otimes T]$ and $V_2 = A \otimes A$. V_1 is isomorphic to a sum of copies of Tate twists of T . Therefore, for example by lemma 3.17, $V_1^G = 0$. This implies that the components of Δ in V_1 are 0. We have $W \oplus V_2 \subset H_{alg}^{2d}(Y \times Y)$, therefore the sum Δ' of the components of Δ in $W \oplus V_2$ is algebraic. By lemma 3.17, $[T \otimes T]^G$ is one dimensional. Therefore $\Delta - \Delta'$ spans it, and this proves algebraicity of ψ in this case.

Let $m > 0$. The Künneth decomposition can be written as

$$H^i(Y^m) = \bigoplus_{i_k, \sigma} \sigma[H^d(Y)^{\otimes j} \otimes H^{i_1}(Y) \otimes \dots \otimes H^{i_{m-1}}(Y)],$$

where each $i_k \neq d$, $dj = i - \sum i_k$ and σ ranges over a set of permutations. When d is even, we refine this further as

$$H^i(Y^m) = \bigoplus_{i_k, j_\ell, \sigma} \sigma[T^{\otimes j_1} \otimes A^{\otimes j_2} \otimes H^{i_1}(Y) \otimes \dots \otimes H^{i_{m-1}}(Y)],$$

for an appropriate set of indices. Let δ be the nonzero rational element of $\wedge^\tau T \subset H^{d\tau}(Y^\tau)$; it is necessarily an antisymmetric Hodge cycle. By our assumptions, we can choose bases $\{\gamma_l^{(i_k)}\}$ for the groups $H^{i_k}(Y)$ (and $\{\gamma'_l\}$ for A if d is even) consisting of algebraic cycles. Then, we have

$$H^i(Y^m) = \bigoplus_{\Gamma, \sigma} \sigma[H^d(Y)^{\otimes j} \times \Gamma]$$

if d is odd, or

$$H^i(Y^m) = \bigoplus_{\Gamma, \sigma} \sigma[T^{\otimes j} \times \Gamma]$$

if d is even, where Γ ranges over products of the γ 's (and γ 's). A Hodge cycle on Y^m can be decomposed into a sum of Hodge cycles from each of the above summands. As an abstract Hodge structure each of these summands is isomorphic to a Tate twist of $H^d(Y)^{\otimes j}$ or $T^{\otimes j}$. As already noted the Hodge cycles here are G -invariant. By lemma 3.17, the G -invariant classes are spanned by the orbit of $\psi \times \psi \times \dots \psi$ and when d is odd. These elements are algebraic by the previous paragraph, and this finishes the proof of theorem 3.15. When d is even, we have additional G -invariant classes lying in the orbit of $\delta \times \psi \dots \psi$, and this accounts for the remaining terms in the statement of theorem 3.16. \square

Corollary 3.18. *Suppose that Y satisfies the assumptions of theorem 3.16. Then the Hodge conjecture holds for arbitrary symmetric powers $S^n Y$.*

Proof. We may assume that $T \neq 0$, or else there is nothing to prove. In this case, $\dim T \geq 2$, since T would contain nontrivial elements of type (p, q) and (q, p) with $p \neq q$.

Consider the map $Y^n \rightarrow S^n Y$. It induces a surjection $H^i(Y^n) \rightarrow H^i(S^n Y)$ by lemma 1.13. Therefore a Hodge cycle γ on $S^n Y$ can be lifted to a Hodge cycle γ' on Y^n . γ' can be expressed as a sum $\alpha + \sigma_1(\delta \times \beta_1) + \dots$ as in theorem 3.16. Since S_n acts trivially on $S^n Y$, the map $H^i(Y^n) \rightarrow H^i(S^n Y)$ factors through the space of coinvariants $H^i(Y^n)_{S_n}$. The image of $\sigma_k(\delta \times \beta_k)$ in $H^i(Y^n)_{S_n}$ must vanish because of the antisymmetry of δ . (More explicitly: $\sigma_k(\delta \times \beta_k)$, $\delta \times \beta_k$ and $(12)(\delta \times \beta_k) = -\delta \times \beta_k$ have the same image.) Therefore γ is represented by the image of α . \square

Corollary 3.19. *Let Y be a smooth projective surface with $q = 0$ and $p_g = 1$ (e.g. a K3 surface) such that $\text{End}_{MHS}(T(Y)) = \mathbb{Q}$, then the Hodge conjecture holds for all symmetric powers of Y .*

Proof. By [Z2, 2.2.1] the hypothesis implies that $\text{Hdg}(T(X)) = \text{SO}(T(X))$. \square

Corollary 3.20. *Suppose that $Z \subseteq \mathbb{P}^N$ is an even dimensional smooth projective variety with a cellular decomposition. If H is a sufficiently general hyperplane section of Z , then the Hodge conjecture holds for all powers of H .*

The precise meaning of “sufficiently general” above is the following: For any Lefschetz pencil $\mathcal{Y} \rightarrow \mathbb{P}^1$ of hyperplane sections of Z , H can be taken to be \mathcal{Y}_t for all but countably many $t \in \mathbb{P}^1(\mathbb{C})$.

Proof. Let $\dim Z = d + 1$. Conditions 1 and 2 of theorem 3.15 hold for any smooth hyperplane section by the weak Lefschetz theorem ($i < d$) and the hard Lefschetz theorem ($i > d$). Suppose that $\mathcal{Y} \rightarrow \mathbb{P}^1$ is a Lefschetz pencil, and let $U \subset \mathbb{P}^1$ be the complement of the set of critical values. Then for any smooth fiber Y_t , $H^d(Y_t) = H^d(Z) \oplus E = E$, where E is the subspace generated by vanishing cycles [K, 5.5]. Since the orthogonal complement $E^\perp = 0$, we can apply the Kazhdan-Margulis theorem [D3] to see that the image of the monodromy representation $\pi_1(U, t) \rightarrow \text{GL}(H^d(Y_t))$ is Zariski dense in $\text{Sp}(H^d(Y_t))$. Then by lemma 3.10 there exists a countable set S such that if $t \notin S$ then Y_t satisfies the third condition of theorem 3.15. \square

It is possible to prove a weaker statement when $\dim Z$ is odd.

Corollary 3.21. *Suppose that $Z \subseteq \mathbb{P}^N$ is a smooth projective variety with a cellular decomposition such that $d = \dim Z - 1$ is even. There exists an integer n_0 such that if H is a sufficiently general hypersurface section of Z of degree $n \geq n_0$, then the Hodge conjecture holds for all symmetric powers $S^m H$.*

Proof. The proof of this is very similar to that of corollary 3.20. Suppose that $\mathcal{Y} \rightarrow \mathbb{P}^1$ is a Lefschetz pencil of hypersurfaces of degree n , and let $U \subset \mathbb{P}^1$ be the complement of the set of critical values. Then for any smooth fiber Y_t , there is an orthogonal decomposition $H^d(Y_t) = A \oplus T$ where $A = H^d(Z)$ and T is the subspace generated by vanishing cycles [K, 5.5]. The monodromy representation $\pi_1(U, t) \rightarrow GL(T)$ is dense in $O(T)$ for n greater than or equal to some $n_0 > 0$ by [V, thm B]. Therefore any finite index subgroup $\Gamma \subseteq \pi_1(U, t)$ contains a subgroup of finite index which is dense in $SO(T)$. Then by lemma 3.10 there exists a countable set S such that if $t \notin S$ then Y_t satisfies the third condition of theorem 3.16, and the first two conditions are automatic. Therefore the result follows from corollary 3.18. \square

4. MODULI OF VECTOR BUNDLES OVER CURVES

When X is a smooth projective curve, let $U_X(n, d)$ be the moduli space of semistable bundles on X of rank n and degree d . It is smooth and projective if n and d are coprime.

Theorem 4.1. *Let X be a smooth projective curve and $M = U_X(n, d)$ with n and d coprime. If the Hodge (respectively generalized Hodge) conjecture holds for all powers X^k , then the Hodge (respectively generalized Hodge) conjecture holds for M .*

Corollary 4.2 (del Baño [db]). *If the Hodge conjecture holds for all powers of $J(X)$, then it holds for M .*

Proof. Proposition 3.8. \square

We record specific instances where this holds:

Corollary 4.3. *If X is*

1. *very general in the moduli space of curves (Biswas-Narasimhan [BN]),*
2. *a curve of genus 2 or 3,*
3. *a curve of prime genus such that the Jacobian is simple, or*
4. *a Fermat curve $x^m + y^m + z^m = 0$ with m prime or less than 21, or,*
5. *a curve admitting a surjection from an $X_1(N)$,*

then the Hodge conjecture holds for M (as above).

Proof. These follow from the results of section 3. For the second case, the Jacobian is either simple or isogenous to a product of elliptic curves. For the last case, X^k satisfies the Hodge conjecture since it is dominated by $X_1(N)^k$. \square

The conclusion of the first part can be strengthened considerably:

Corollary 4.4. *If X is very general in the moduli space of curves, then the generalized Hodge conjecture holds for M .*

Proof. This follows from corollary 3.7 and proposition 3.8. \square

We give two proofs of the theorem. The first is fairly elementary, while the second is easier to generalize. Let E be a vector bundle on X and $Q_n(E)$ denote the “Quot” scheme parameterizing coherent subsheaves of $F \subset E$ such that E/F is a torsion sheaf of length n . By [Gr] $Q_n(E)$ exists and is projective, and it can be seen to be smooth since X is a curve. When $E = O_X$, this is just the space of anti-effective divisors of degree $-n$.

First proof. Choose a divisor D with $\deg D \gg 0$ (the precise requirements will be given below). Let $E = O(D)^{\oplus n}$, $m = n \deg(D) - d$, and $Q = Q_m(E)$. A point of Q is given by an abstract vector bundle F on X of rank n and degree d together with an embedding $F \subset E$. Let Q^s denote the open subset which parameterizes pairs $(F \subset E)$ with F stable. There is an obvious map $\pi : Q^s \rightarrow M$. Choose $\deg D$ sufficiently large, so that $F^*(D)$ is globally generated and $H^1(F^*(D)) = 0$ for any stable vector bundle $F \in M$. Thus any $F \in M$ can be embedded into E which implies that $\pi : Q^s \rightarrow M$ is surjective. Let \mathcal{F} denote a Poincaré bundle on $X \times M$. By our assumptions, $V = p_{X*} \mathcal{H}om(\mathcal{F}, p_M^* E)$ is locally free and commutes with base change, where p_X, p_M denote the projections of $X \times M$ to its factors. Let $\pi' : \mathbb{P}(V^*) \rightarrow M$ denote the bundle of lines in the fibers of V . We have an embedding $Q^s \hookrightarrow \mathbb{P}(V^*)$, which sends the point $(F \subset E)$ to the line generated by the corresponding element of $\mathcal{H}om(F, E)$. The codimension of the complement $Z = \mathbb{P}(V^*) - Q^s$ is greater than or equal to $\deg D$ [BGL, 8.2]. Let us assume that $\deg D > (\dim M + 1)/2$.

The torus $T = (\mathbb{C}^*)^n$ acts on $E = O(D)^{\oplus n}$ by homotheties on each factor, and this induces an action on Q [BGL]. The components of Q^T are products of symmetric powers of X [loc. cit.]. In particular, the (generalized) Hodge conjecture holds for these components by corollary 1.15. Therefore the (generalized) Hodge conjecture holds for Q by lemma 2.4, and consequently for Q^s by proposition 1.5. We have an exact sequence

$$H_{2q-i}(Z) \rightarrow H_{2q-i}(\mathbb{P}(V^*)) \rightarrow H_{2q-i}(Q^s) \rightarrow \dots$$

where $q = \dim Q$. By our assumptions,

$$H_{2q-i}(\mathbb{P}(V^*)) \cong H_{2q-i}(Q^s)$$

if $i \leq 2 \dim M$. Therefore $GHC(H_{2q-i}(\mathbb{P}(V^*)), 2(2q-i))$ (or $GHC(H_{2q-i}(\mathbb{P}(V^*)), *)$) holds, or equivalently $GHC(H^i(\mathbb{P}(V^*)), 2i)$ (or $GHC(H^i(\mathbb{P}(V^*)), *)$) holds for $i \leq 2 \dim M$. The theorem now follows from corollary 1.4. \square

Embedded in this argument is a proof that $Q_n(E)$ satisfies the Hodge conjecture for $E = O(D)^n$. This is true more generally:

Proposition 4.5. *If the Hodge (respectively generalized Hodge) conjecture holds for all powers X^k , then the Hodge (respectively generalized Hodge) conjecture holds for $Q_n(E)$ for any n and vector bundle E .*

The proof will be deferred to the next section.

Suppose Y and N are compact oriented manifolds and $c \in H^*(Y \times N, \mathbb{Q})$. We can decompose c as

$$c = \sum p_Y^* d_i \cup p_N^* e_i = \sum d_i \times e_i$$

by the Künneth formula. The e_i will be called the Künneth components of c along N . By Poincaré duality c can be identified with a homomorphism $H^*(Y) \rightarrow$

$H^*(M)$. Explicitly, this is

$$c(d') = \sum \int_Y d_i \cup d' e_i$$

Thus the image of c is contained in, and in fact equal to, the span of the Künneth components. The second proof will be based on the following:

Proposition 4.6. *Suppose that Y and N are smooth projective varieties such that Y^k satisfies the Hodge (respectively generalized Hodge) conjecture for all $k \leq \dim N$. In addition, assume that there exists a finite collection of algebraic correspondences on $Y \times N$ such their Künneth components generate the cohomology ring $H^*(N, \mathbb{Q})$. Then the Hodge (respectively generalized Hodge) conjecture holds for N .*

Proof. Let $c_{j,i} \in H^{2i}(Y \times N)$ denote the cohomology classes of the above algebraic correspondences. Taking exterior products, we get correspondences $c_{j_1, i_1}(E) \times \dots \times c_{j_n, i_n}(E)$ on $Y^n \times N^n$. These can be pulled back along the diagonal map $N \hookrightarrow N^n$ to get correspondences $C_{J,I} = C_{j_1, \dots, j_n, i_1 \dots i_n}$ on $Y^n \times N$. which induce morphisms (as in corollary 1.2)

$$H^*(Y^n)(-(i_1 + \dots + i_n - n)) \rightarrow H^{*+2(i_1 + \dots + i_n - n)}(N).$$

Let

$$A : \oplus_{J,I} H^*(Y^*)(*) \rightarrow H^a(N)$$

be the sum of the $C_{J,I}$ maps as $I = (i_1, \dots, i_n)$ varies over all finite sequences with $\sum i_k \leq a/2$ and $i_k > 0$. The hypothesis implies that $H^a(N)$ is spanned by monomials in the Künneth components of the $c_{j,i}$, and this is equivalent to the surjectivity of the map A . Therefore corollary 1.4 finishes the proof. \square

Second proof of theorem 4.1. Let E be a Poincaré bundle on $X \times M$. The Chern classes give correspondences $c_i(E) \in H_{alg}^{2i}(X \times M)$. The Künneth components of these classes generate the cohomology ring $H^*(M)$ by a theorem of Atiyah and Bott [AB, 9.11] or more specifically Beauville's version of this theorem [Be2]. Thus we can apply the previous proposition. \square

Remark 4.7. *This proof actually gives a slightly stronger result that the (generalized) Hodge conjecture holds for M if it holds for all powers X^k with $k \leq \dim M$.*

5. MODULI OF SHEAVES OVER SURFACES

For surfaces the analogous results are a bit more elusive. We begin with the moduli space of ideals of zero dimensional subschemes, i.e. the Hilbert scheme of points. We review some basic facts about these spaces; further details can be found in [G]. Let X be a smooth projective variety. For each integer $n > 0$, let $X^{(n)} = S^n X$ be the n th symmetric power, and let $X^{[n]}$ be the Hilbert scheme of zero dimensional subschemes of X of length n . There are canonical morphisms $p : X^n \rightarrow X^{(n)}$ and $\psi : X_{red}^{[n]} \rightarrow X^{(n)}$. The map ψ , called the Hilbert-Chow morphism, is birational.

Theorem 5.1 (Forgarty). *If $\dim X = 2$, then $X^{[n]}$ is smooth (and projective) for each n . It follows that $\psi : X^{[n]} \rightarrow X^{(n)}$ is a resolution of singularities.*

These spaces have a natural stratification. Given a partition $\lambda = (n_1, n_2, \dots, n_k)$ of n (i.e. a nonstrictly decreasing sequence of positive integers summing to n), let

$$X_\lambda^{(n)} = \{p(x_1, \dots, x_n) \mid x_1 = x_2 = \dots = x_{n_1} \neq x_{n_1+1} = \dots = x_{n_1+n_2} \neq \dots\}$$

and let

$$X_\lambda^{[n]} = \psi^{-1} X_\lambda^{(n)}.$$

These are locally closed subsets of $X^{(n)}$ and $X^{[n]}$ which will be regarded as subschemes with reduced structure. The scheme $X_{(n)}^{[n]}$ parameterizes 0-dimensional subschemes with support at a single point. There is a morphism $\pi_n : X_{(n)}^{[n]} \rightarrow X$ which sends a subscheme to its support.

Lemma 5.2. ([G, 2.1.4, 2.2.4]) *π_n is a locally trivial fiber bundle. When $\dim X = 2$, the fiber is smooth, projective and has an cellular decomposition.*

Let $U_k \subset X^k$ be the open subset of k -tuples with distinct components. For a partition $\lambda = (n_1, \dots, n_k)$ of n , define

$$X_\lambda^{<n>} = U_k \times_{X^k} \prod_{i=1}^k X_{(n_i)}^{[n_i]}$$

Lemma 5.3. ([G, 2.3.3]) *$X_\lambda^{[n]}$ is a quotient of $X_\lambda^{<n>}$ by a finite group.*

Theorem 5.4. *Let X be a smooth projective surface such that all powers X^k , with $k \leq n$, satisfy the Hodge (respectively generalized Hodge) conjecture. Then $X^{[n]}$ satisfies the Hodge (respectively generalized Hodge) conjecture.*

Proof. Let $\lambda = (n_1, \dots, n_k)$ be a partition of n . By lemma 5.2, the π_{n_i} are locally trivial fiber bundles such that the fibers are smooth and projective with cellular decompositions. Therefore $\prod X_{(n_i)}^{[n_i]}$ is a fiber bundle over X^k with these properties. Then lemma 1.9 and remark 1.11 imply that $\prod X_{(n_i)}^{[n_i]}$ satisfies the (generalized) Hodge conjecture. Since $X_\lambda^{<n>}$ is an open subset of this space, it also satisfies the (generalized) Hodge conjecture by proposition 1.5.

Lemma 5.3 implies that $X_\lambda^{[n]}$ has finite quotient singularities and that $X_\lambda^{<n>}$ is maps onto it. Corollary 1.15, and the remarks preceding it, yields the (generalized) Hodge conjecture for $X_\lambda^{[n]}$. Corollary 1.7 finishes the proof. \square

Corollary 5.5. *When X and k are as in proposition 3.14, $X^{[k]}$ satisfies the Hodge conjecture.*

Proposition 4.5 can be proved by modifying the above argument. We will sketch this.

Proof of proposition 4.5. There is an analogue of the Hilbert-Chow morphism $\psi : Q_n(E) \rightarrow S^n X = X^{(n)}$ which sends

$$(F \subset E) \mapsto \sum_{p \in X} \text{length}(E_p/F_p) p$$

$X^{(n)}$ can be stratified as above. The strata satisfy the Hodge conjecture since their closures are dominated by powers of X . The restriction of ψ to these strata are locally trivial fiber bundles where the fibers are products of Grassmanians. Thus

the preimages of the strata under ψ satisfy the Hodge conjecture. We can now conclude the proof by corollary 1.7. \square

Let us turn to the higher rank case. Given a smooth projective surface X , choose a polarization H and elements $r \in \mathbb{N}$, $c_i \in H^{2i}(X, \mathbb{Z})$. Let

$$ch = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2)$$

be the ‘‘Chern character’’, and let K , $[X]$ and $td(X)$ respectively denote the canonical, fundamental and Todd classes of X . Call a class $c \in H^*(X, \mathbb{Z})$ primitive if it is not multiple of a class other than $\pm c$. Let $M_X(r, c_1, c_2, H)$ be the moduli space of torsion free sheaves on X of rank r with Chern classes given by c_i which are H -semistable in the sense of Gieseker-Maruyama [Gi] [Ma]. This is a projective scheme of finite type. The open set of stable sheaves $M_X^s(r, c_1, c_2, H)$ tends to have more manageable local properties. In particular, it is known to be smooth if $-K_X \cdot H < 0$ [Ma, 6.7.3], or if X is abelian or K3 [Mk1]. In ideal cases, the stable locus coincides with the whole moduli space. Part of the standard folklore is:

Lemma 5.6. *If $\gcd(r, c_1 \cdot H, ch \cdot td(X)) = 1$, then $M_X^s(r, c_1, c_2, H) = M_X(r, c_1, c_2, H)$*

Proof. If E lies in $M - M^s$, then there exists subsheaf $F \subset E$ with $rk(F) = s < r$ such that

$$\begin{aligned} \frac{\chi(E(n))}{r} - \frac{\chi(F(n))}{s} &= \left(\frac{c_1(E) \cdot H}{r} - \frac{c_1(F) \cdot H}{s} \right) n \\ &\quad + \left(\frac{ch(E) \cdot td(X)}{r} - \frac{ch(F) \cdot td(X)}{s} \right) \\ &= 0 \end{aligned}$$

for all $n \gg 0$. This contradicts the gcd condition. \square

Suppose X is an abelian (respectively K3) surface such that ch (respectively $ch(1 + [X])$) is primitive. Then $M_X^s(r, c_1, c_2, H) = M_X(r, c_1, c_2, H)$ provided that H sufficiently general (this can be deduced from [Y]).

Theorem 5.7. *Let $X, H \dots$ be as above, and let $M = M_X(r, c_1, c_2, H)$ and $M^s = M_X^s(r, c_1, c_2, H)$. Then*

1. *The generalized Hodge conjecture holds for every component of M if X is rational, $-K \cdot H < 0$, and the hypothesis of lemma 5.6 holds.*
2. *The Hodge (respectively generalized Hodge) conjecture holds for every component of M if $-K \cdot H < 0$, the hypothesis of lemma 5.6 holds and X is minimal ruled over a curve C all of whose powers satisfy the Hodge (respectively generalized Hodge) conjecture.*
3. *The Hodge (respectively generalized Hodge) conjecture holds for every component of M if $M = M^s$, X is abelian or K3, and all powers of X satisfy the Hodge (respectively generalized Hodge) conjecture.*

Proof. The result will be reduced to proposition 4.6 in each of the above cases. Note that X^k satisfies the (generalized) Hodge conjecture for all k either by hypothesis or by proposition 3.14. Under the assumptions of (1) or (2), Maruyama [Ma] has shown that M that there is a universal sheaf \mathcal{E} on $X \times M$ and that $Ext_X^2(E_1, E_2) = 0$ for any two sheaves in $E_i \in M$. Therefore, Beauville’s theorem [Be2] applies to show

that $H^*(M)$ is generated by the Künneth components of $c_i(\mathcal{E})$ as an algebra. The result follows by proposition 4.6. This finishes the proof in cases (1) and (2).

In case (3), we can apply the main theorem of Markman [Mrk] to see that $H^*(M)$ is generated by the Künneth components of Chern classes of a quasi-universal sheaf \mathcal{E} on $X \times M$. \square

Corollary 5.8. *The Hodge conjecture holds for M when X is an abelian surface or a K3 surface satisfying the conditions of proposition 3.14 (e) or (f).*

Proof. Follows from proposition 3.14. \square

Corollary 5.9. *The generalized Hodge conjecture holds for M when X is a product of two elliptic curves without complex multiplication or X is a simple abelian surface of type I or II.*

Proof. This follows from corollary 3.6 \square

6. ARITHMETIC ANALOGUES

Let k be a field which is finitely generated over a prime field. Let \bar{k} denote the separable closure and $G = \text{Gal}(\bar{k}/k)$ the Galois group. Choose a prime $l \neq \text{char } k$. Given a variety X defined over k , let $\bar{X} = X \times_{\text{spec } k} \text{spec } \bar{k}$. Then the étale cohomology group $H_{\text{et}}^i(\bar{X}, \mathbb{Q}_l)$ is a \mathbb{Q}_l -vector space with a continuous G -action. Tate twisting in this context amounts twisting by a power of the cyclotomic character (see [K] for a rapid introduction to these ideas). When X is smooth and projective we refer to an invariant in $H_{\text{et}}^{2i}(\bar{X}, \mathbb{Q}_l(i))^G$ as a Tate cycle. The fundamental class of a subvariety defined over k is a Tate cycle. The Tate conjecture [T1, T3] claims conversely that the space of Tate cycles is spanned by these fundamental classes. This can be viewed as an analogue of the Hodge conjecture. To make this analogy clearer, note that the space of Hodge cycles on the $2i$ th cohomology of a complex smooth projective variety is isomorphic to $\text{Hom}_{MHS}(\mathbb{Q}, H^{2i}(X, \mathbb{Q})(i))$. Similarly the space of Tate cycles is $\text{Hom}_G(\mathbb{Q}_l, H_{\text{et}}^{2i}(\bar{X}, \mathbb{Q}_l(i)))$.

Deligne [DMOS, I] has proposed a variant of the Hodge conjecture that says roughly that the property of being a Hodge cycle should be invariant under the automorphism group of \mathbb{C} . We will give the precise formulation in a manner which generalizes easily. Suppose that k is a finitely generated field of characteristic 0. Jannsen [J, §2] has constructed the abelian category MR_k of mixed realizations. An object in this category is a collection of the following data:

1. A bifiltered finite dimensional k -vector space (H_{dR}, F, W)
2. A filtered finite dimensional \mathbb{Q}_l vector space with a continuous G -action (H_l, W) for each prime l .
3. A \mathbb{Q} -mixed Hodge structure H_σ , (H_σ, F, W) for each embedding $\sigma : \bar{k} \rightarrow \mathbb{C}$.
4. Comparison isomorphisms $H_\sigma \otimes \mathbb{Q}_l \cong H_l$ and $H_{dR} \otimes \mathbb{C} \cong H_\sigma \otimes \mathbb{C}$ respecting the filtrations.

The morphisms are constructed so that the obvious projections of MR_k are functors. In particular, for each embedding $\bar{\sigma} : \bar{k} \rightarrow \mathbb{C}$, there is a functor from $\Phi_{\bar{\sigma}} : MR_k \rightarrow MHS$. For example, if X is a smooth projective variety over k , the collection of de Rham, l -adic and analytic cohomologies

$$(H_{dR}^i(X), H_{\text{et}}^i(\bar{X}, \mathbb{Q}_l), H^i(\bar{X} \times_{\sigma} \text{spec } \mathbb{C}, \mathbb{Q})),$$

along with the comparison isomorphisms and appropriate filtrations provides an example of an object $H_{AH}^i(X, 0) \in MR_k$. More generally, Jannsen [J, 6.11.1] has constructed a (co)homology theory $H_{AH}^*(X, j), H_*^{AH}(X, j)$ from the category of k -varieties X to MR_k such that

$$\begin{aligned}\Phi_\sigma(H_{AH}^*(X, j)) &= H^*(\bar{X} \times_\sigma \text{spec } \mathbb{C}, \mathbb{Q}(j)) \\ \Phi_\sigma(H_*^{AH}(X, j)) &= H_*(\bar{X} \times_\sigma \text{spec } \mathbb{C}, \mathbb{Q}(j)).\end{aligned}$$

Given an object $H \in MR_k$, the space of absolute Hodge cycles

$$\Gamma(H) = \{(\alpha_{dR}, \dots) \mid \text{the components are compatible, } \alpha_{dR} \in F^0 H_{dR} \cap W_0 H_{dR}\}$$

If X is a smooth projective variety over k , then each component α_σ of $\alpha \in \Gamma(H_{AH}^{2i}(X, i))$ is a Hodge cycle. Deligne's conjecture, which we will refer to as the absoluteness conjecture, is that for each σ , any Hodge cycle $H^{2i}(\bar{X} \times_\sigma \text{spec } \mathbb{C}, \mathbb{Q}(i))$ arises as α_σ for some absolute Hodge cycles α . This conjecture would follow from the Hodge conjecture since the collection of fundamental classes of an algebraic cycle defined over k for the above cohomology theories is an absolute Hodge cycle.

We need the analogues of the results of section 1.

Lemma 6.1. *Let X and Y be smooth projective k -varieties and suppose that $f : H_{et}^{2i}(\bar{X}, \mathbb{Q}_l(i)) \rightarrow H_{et}^{2j}(\bar{Y}, \mathbb{Q}_l(j))$ is a surjective morphism induced by a correspondence defined over k . If the Tate conjecture holds for X and if $\text{char } k = 0$ or $H^*(\bar{X})$ is a semisimple G -module, then the Tate conjecture holds for Y . If $\text{char } k = 0$ and the absoluteness conjecture holds for X , then it holds for Y .*

Proof. If the map $f : H^*(\bar{X}) \rightarrow H^*(\bar{Y})$ of Galois modules splits, then any Tate cycle α in $H^*(\bar{Y})$ can be lifted to a Tate cycle β in $H^*(\bar{X})$. β would be algebraic if Tate's conjecture held for X , therefore α is also algebraic. The splitting of f is immediate when $H^*(\bar{X})$ is semisimple. When $\text{char } k = 0$, the splitting follows from [J, 1.2].

Similarly by corollary 1.4, a Hodge cycle α in $H^*(\bar{Y} \times_\sigma \text{spec } \mathbb{C})$ can be lifted to a Hodge cycle β in $H^*(\bar{X} \times_\sigma \text{spec } \mathbb{C})$. Assuming the absoluteness conjecture for X , β and hence α , would necessarily extend an absolute Hodge cycle. \square

Remark 6.2. *It is conjectured that $H^*(\bar{X})$ is always semisimple. However this has been established in only very special cases [Fa, T2, Z1].*

Jannsen [J, 7.3] has also formulated a version of Tate's conjecture for singular quasiprojective varieties using Borel-Moore étale homology which can be defined as dual to compactly supported étale cohomology as above. Given such a variety X , the fundamental class of any i dimensional k -subvariety lies in $H_{2i}^{et}(\bar{X}, \mathbb{Q}_l(-i))^G$. We say that the Tate conjecture holds for X if $H_{2*}^{et}(\bar{X}, \mathbb{Q}_l(*))^G$ is spanned by these classes. The absoluteness conjecture can be extended in a similar fashion: namely, that Hodge cycles in $H_{2i}(\bar{X} \times_\sigma \text{spec } \mathbb{C}, \mathbb{Q}(-i))$ lift to absolute Hodge cycles $\Gamma(H_*^{AH}(X, -i))$.

Proposition 6.3 (Jannsen[J]). *If $\text{char } k = 0$, the Tate conjecture holds for a k -variety U if it holds for a desingularization of a compactification of U .*

Lemma 6.4. *Suppose that k is an algebraically closed field of characteristic 0. The absoluteness conjecture holds for a k -variety U if it holds for a desingularization of a compactification of U .*

Proof. Fix an embedding $\sigma : k \rightarrow \mathbb{C}$, and let $Y_\sigma = \bar{Y} \times_\sigma \text{spec } \mathbb{C}$. Let \tilde{X} be a desingularization of a compactification of U satisfying the absoluteness conjecture. Thus the map from the space of absolute Hodge to Hodge cycles

$$\Gamma H_{2i}^{AH}(\tilde{X}, i) \rightarrow H_{2i}^{Hodge}(\tilde{X}_\sigma)$$

is surjective. Let $X \subseteq \tilde{X}$ be the preimage of U . Then the composition of

$$H_{2i}(\tilde{X}_\sigma, \mathbb{Q}(i)) \rightarrow H_{2i}(X_\sigma, \mathbb{Q}(i)) \rightarrow H_{2i}(U_\sigma, \mathbb{Q}(i))$$

induces a surjection

$$H_{2i}^{Hodge}(\tilde{X}_\sigma, \mathbb{Q}(i)) \rightarrow H_{2i}^{Hodge}(U_\sigma, \mathbb{Q}(i))$$

(see [J, pp 113-114]). Consequently

$$\Gamma H_{2i}^{AH}(U, i) \rightarrow H_{2i}^{Hodge}(U_\sigma, i)$$

is surjective since it factors through the composition

$$\Gamma H_{2i}^{AH}(\tilde{X}, \mathbb{Q}(i)) \rightarrow H_{2i}^{Hodge}(U_\sigma, \mathbb{Q}(i)).$$

□

Lemma 6.5. *Suppose that $\text{char } k = 0$. Let $Z \subset X$ be a closed subset of a projective variety X , and let $U = X - Z$. The Tate conjecture (respectively the absoluteness conjecture) holds for X if it holds for U and Z .*

Proof. There is an exact sequence of Galois modules

$$H_{2k}^{et}(\bar{Z}, \mathbb{Q}_l) \rightarrow H_{2k}^{et}(\bar{X}, \mathbb{Q}_l) \rightarrow H_{2k}^{et}(\bar{U}, \mathbb{Q}_l).$$

From weight considerations, the image I of $H_{2k}^{et}(\bar{Z}, \mathbb{Q}_l)$ in $H_{2k}^{et}(\bar{X}, \mathbb{Q}_l)$ coincides with the image of $H_{2k}^{et}(\bar{Z}, \mathbb{Q}_l)$ for any desingularization $\tilde{Z} \rightarrow Z$. By [J, 1.2], I possesses a complement in the category of Galois modules. It follows that there is an exact sequence of Tate cycles

$$H_{2k}^{et}(\bar{Z}, \mathbb{Q}_l(-k))^G \rightarrow H_{2k}^{et}(\bar{X}, \mathbb{Q}_l(-k))^G \rightarrow H_{2k}^{et}(\bar{U}, \mathbb{Q}_l(-k))^G$$

Likewise by lemma 1.6, there is an exact sequence of absolute Hodge cycles

$$\Gamma(H_{2k}(Z, -k)) \rightarrow \Gamma(H_{2k}(X, -k)) \rightarrow \Gamma(H_{2k}(U, -k)).$$

The rest of the argument proceeds exactly as in the proof of lemma 1.6. □

From this lemma, we can deduce the analogue of corollary 1.7 for the Tate and Deligne conjectures when k has characteristic 0.

Lemma 6.6. *Suppose $\text{char } k = 0$. Let $f : X \rightarrow Y$ be a morphism of smooth k -varieties which is a Zariski locally trivial fiber bundle with fiber F . Suppose that F is smooth and that $H_*^{et}(\bar{F}, \mathbb{Q}_l)$ is spanned by algebraic cycles. Then the Tate (or absoluteness) conjecture holds for X if it holds for Y .*

The proof of this is virtually identical to the proof of lemma 1.9 (the Leray-Hirsch theorem for etale cohomology can be deduced by the comparison theorem).

Milne [Mi, sect. 4] has proven a version of theorem 3.1 for the Tate conjecture. We note a special case.

Proposition 6.7. *If X is a polarized abelian variety over k such that the image of G is Zariski dense in $GSp(H_{et}^1(\bar{X}, \mathbb{Q}_l))$, then the Tate conjecture holds for all powers of X .*

Proof. In the terminology of [Mi], the Zariski closure of the image of G is a subgroup of the Tate group which is contained in the Lefschetz group $\subseteq GSp(H_{et}^1(\bar{X}, \mathbb{Q}_l))$. When equality holds, Tate's conjecture holds for X and its powers. \square

Corollary 6.8. *If k is a number field, $End(\bar{X}) = \mathbb{Z}$ and $\dim X$ is either odd or equal to 2 or 6, then the Tate conjecture holds for all powers of X .*

Proof. Under these conditions G is dense in $GSp(H^1(X))$ by a theorem of Serre [Se, 2.28] (see also [C, 6.1]). \square

Deligne [DMOS, I,II] has proven that the absoluteness conjecture holds for certain special classes of varieties.

Theorem 6.9 (Deligne). *Let X be an abelian variety, a product of K3 surfaces, or a product of Fermat hypersurfaces. Then the absoluteness conjecture holds.*

Corollary 6.10. *Let X be smooth projective curve, then the absoluteness conjecture holds for all powers of X .*

Proof. The argument is similar to the proof of proposition 3.8. \square

The following are analogues of proposition 4.6 and theorem 4.1.

Proposition 6.11. *Suppose that Y and N are smooth projective varieties over k such that there exists a finite collection of algebraic correspondences on $Y \times N$ such that their Künneth components generate the cohomology ring $H_{et}^*(\bar{N}, \mathbb{Q}_l)$. Assume that either $H_{et}^1(\bar{Y}, \mathbb{Q}_l)$ is semisimple or $\text{char } k = 0$. If such that Y^m satisfies the Tate conjecture for all $m \leq \dim N$, then the Tate conjecture holds for N . If k is an algebraically closed field of characteristic 0, then the absoluteness conjecture for Y^m for all $m \leq \dim N$ implies the absoluteness conjecture for N .*

Proof. As in the proof of proposition 4.6 we get a surjection

$$A : \oplus_{J,I} H^*(\bar{Y}^*)(*) \rightarrow H^a(\bar{N})$$

induced by a correspondence, where H^* is taken to be l -adic cohomology or H_{AH}^* . An appeal to lemma 6.1 finishes the proof. \square

Theorem 6.12. *Let X be a smooth projective curve over k , and let $M = U_X(n, d)$ with n and d coprime. Assume that either $H_{et}^1(X, \mathbb{Q}_l)$ is semisimple or $\text{char } k = 0$. Then the Tate conjecture holds for M if it holds for all powers of X . If k is an algebraically closed field of characteristic 0, then the absoluteness conjecture holds for M .*

Remark 6.13. *The part of the theorem concerning the Tate conjecture in characteristic 0 is due to del Baño [db].*

Proof. The argument is identical to second proof of theorem 4.1 with proposition 6.11 replacing proposition 6.11. (Note that Beauville's proof [Be2] is algebraic and is valid in positive characteristic provided etale cohomology is used in place of singular cohomology.) \square

Corollary 6.14. *The Tate conjecture holds for M if the hypotheses of corollary 6.8 are satisfied for k and $J(X)$.*

Proof. As in the proof of proposition 3.8, one gets a surjection $H_{et}^*(\bar{J}(X)^m) \rightarrow H_{et}^*(\bar{X}^m)$. Lemma 6.1 implies the Tate conjecture for X^m . \square

The proof of theorem 5.4 can be modified to yield:

Theorem 6.15. *Suppose that $\text{char } k = 0$. Let X be a smooth projective surface such that all powers X^m , with $m \leq n$, satisfy the Tate conjecture (respectively the absoluteness conjecture). Then $X^{[n]}$ satisfies the Tate conjecture (respectively the absoluteness conjecture).*

Corollary 6.16. *Suppose that $\text{char } k = 0$. Let X be a smooth projective surface over k with Kodaira dimension $\kappa(X) \leq 0$, then the absoluteness conjecture holds for $X^{[n]}$ for any n .*

Proof. First note that when X is a product of \mathbb{P}^1 and a smooth projective curve, the absoluteness conjecture holds for all powers of X by corollary 6.10 and lemma 6.6. Then arguing as in the proof of 3.14, we see that if there is a smooth projective surface Y , all of whose powers satisfy the absoluteness conjecture, and a dominant rational map $Y \dashrightarrow X$, then the absoluteness conjecture holds for all powers of X as well.

If X is a smooth surface of Kodaira dimension zero, it is a:

1. rational or ruled,
2. abelian or bielliptic, or
3. K3 or Enriques

surface [Be1]. Then there exists a dominant map $Y \dashrightarrow X$ where Y is a product of \mathbb{P}^1 and curve in case 1, an abelian surface in case 2, or a K3 surface in case 3. This finishes the proof. \square

We have an analogue of theorem 5.7. For simplicity, we state the most interesting part.

Theorem 6.17. *Suppose that $\text{char } k = 0$. Let X be an abelian or K3 surface defined over k , and assume that classes r, c_1, c_2, H have been chosen so that $M = M_X(r, c_1, c_2, H) = M_X^s(r, c_1, c_2, H)$. If the Tate conjecture holds for all powers of X , then it holds for M . If k is algebraically closed, then the absoluteness conjecture holds for M .*

Proof. The proof is identical to the proof of theorem 5.7 with proposition 6.11 in the place of proposition 4.6. \square

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